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Congruences of Fluids in a Finslerian Anisotropic Space-Time[1]

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We derive the generalized Raychauduri equation concepts of expansion, shear and vorticity. We give the Ricci tensor of a constant-curvature Randers–Finsler space metric whose first term is the Robertson–Walker metric.

KEY WORDS: fluids; congruences; Finsler geometry; field equations.

1. INTRODUCTION

During the last year, observational investigations of the increased values of anisotropy of microwave cosmic background radiation (Bennet *et al.*, 2003; Patridge, 1995) suggest the introduction of an anisotropic metric structure in the underlying geometry of space-time. A candidate geometry for the study of generalized field equations with respect to the density and pressure of fluids moving in anisotropic gravitational fields, is Finsler geometry.

Many researchers have studied properties of the gravitational field and of space-time in the framework of this geometry. We indicatively mention (Asanov, 1985; Asanov and Stavrinos, 1991; Balan and Stavrinos, 2002; Beil, 1989, 2003; Ikeda, 1995; Ishikawa, 1981; Miron and Anastasiei, 1987; Stavrinos, 2002; Stavrinos and Diakogiannis, 2004; Stavrinos and Ikeda, 2004; Vacaru, 2001; Vacaru and Stavrinos, 2002).

In the next section we shall consider the concept of expansion, shear and vorticity of time-like flows as these are defined in the Finslerian context and we shall use them to derive the generalized Raychauduri equation. This equation plays an important role in the Riemannian prototype of general relativity (Ellis *et al.*, 1990; Hawking and Ellis, 1973).

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The derivation of the field equations of fluids in the Finslerian space-time can be considered for a possible test of Finsler geometry in connection with the observational values of microwave background radiation, given that a special Finsler–Randers type space of constant curvature is used.

Under these circumstances the Robertson–Walker metric is no longer valid. However, this metric can constitute a part of an anisotropic Finslerian metric that we introduce in the form

$$L(x, y) = L_{\text{R-W}}(x, y) + \phi(x)\bar{u}^a y_a \tag{1}$$

where

$$L_{\text{R-W}}(x, y) = \frac{1}{2}\{\dot{t}^2 - R^2(t)[(1 - kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2]\}^{1/2}$$

is the Lagrangian of the Robertson–Walker metric, $\bar{u}^a$ represents a unit vector which expresses the observed anisotropy of the microwave background radiation, $\phi(x)$ is a scalar function and $y^a = \frac{dx^a}{dt}$ denotes a direction in the space.

A general form of the metric (1) has been defined previously in (Stavrinos and Diakogiannis, 2004), where the first term of (1) is substituted by the pseudo-Riemannian Lagrangian metric $L_{\text{R}} = \sqrt{g_{ij} y^i y^j}$ into studying the geometrical properties of an anisotropic direction-dependent Finslerian space-time.

2. FINSLERIAN CONGRUENCES OF ANISOTROPIC FLOWS

A Finsler space is constructed by a differentiable manifold and a fundamental smooth metric function $F(x, y)$ on its tangent bundle TM which depends on the variables, $x \in M$ of position and $y = \frac{dx}{dt}$ of direction in which F is homogeneous of first degree with respect to y (Rund, 1959; Miron, 1987).

Suppose $(F^4, g_{ij}(x, y))$ is a four dimensional differentiable manifold and $g_{ij}(x, y)$ the anisotropic Finslerian metric is assumed to have signature $(+, -, -, -)$ for any (x, y).

The motion of a particle in a Finslerian space-time F^4 is described by a pair (x, V) where $x \in F^4$ and $V = \frac{dx}{d\tau}$ the 4-velocity of the particle (τ is proper time) which represents the tangent of its world-line expressing the motion of typical observers in the Finslerian anisotropic universe.

A *smooth congruence* in an open coordinate neighborhood U of F^4 can be represented by a preferred family of world lines (time-like curves) such that through each couple $(x, V) \in U$ there passes precisely one curve in this family in which V is the tangent vector of this curve to that point x. This consideration is analogous to the Riemannian context.

The metric of Finslerian space-time is described by the relation

$$ds^2 = F^2(x, y) = g_{ij} y^i y^j$$

The time-like, null and space-like curves can be defined in the Finslerian framework by the following relations (Ishikawa, 1981)

$$\begin{aligned} \text{time-like} \quad & g_{ij}(x, y)V^i V^j > 0 \\ \text{null-like} \quad & g_{ij}(x, y)V^i V^j = 0 \\ \text{space-like} \quad & g_{ij}(x, y)V^i V^j < 0 \end{aligned} \tag{2}$$

In the following we assume Finslerian fluid congruences that the matter flow lines of the fluid are time-like geodesics and are parameterized by the proper time τ so that a vector field $V^i(x)$ of tangents is normalized to the unit length $V^i = \frac{y^i}{F}$. We denote by $\Lambda_{ik} = V_{i;k}$ the Finslerian δ-covariant derivative with respect to the direction of $V(x)$ (Rund, 1959).

We notice that Λ_{ik} belongs to the normal subspace of the tangent space

$$\Lambda_{ik} V^k = 0, \quad \Lambda_{ik} V^i = 0 \tag{3}$$

These are followed because of geodesic condition and the relation of normalization that means Λ_{ik} is a "spatial" vector.

A physical and geometrical interpretation can be given if we consider a smooth one parameter $C_s(\tau)$ congruence of Finslerian geodesics. Because of the equation of geodesics deviation (Rund, 1959), the deviation vector z^i provide us the separation from a geodesics C_0 to a nearby one of the family.

From the condition

$$\mathcal{L}_V z^i = 0 \tag{4}$$

we get

$$z^i_{;\ell} V^\ell = V^i_{;\ell} z^\ell - \frac{\partial z^i}{\partial y^n} \left(V^n_{;\ell} y^\ell \right) = \Lambda^i_\ell z^\ell - \frac{\partial z^i}{\partial y^n} \Lambda^n_\ell y^\ell \tag{5}$$

where $\mathcal{L}$ represents the Finslerian Lie variation.

The tensor field Λ^j_i measures the change of z^i to be parallel—transported along—a Finslerian stream line. From a physical point of view an observer moving along the geodesic C_0 would find the adjacent geodesics surrounding him to be stretched and rotated by the field Λ^j_i. We write down the angular metric h_{ij} in the Finslerian framework

$$h_{ij} = g_{ij} - V_i V_j$$

where V^i is the unit tangent vector. This tensor has the property

$$h_{ij} V^i = 0. \tag{6}$$

Using the δ-differentiation in the direction of $V^i(x)$ for a congruence of fluid lines (not necessarily geodesics) we define the expansion, vorticity and the shear

(Asanov, 1985) by the following forms:

$$\widetilde{\Theta} = \Lambda_{ij}h^{ij} = V^i_{|i} - C^i_{im}\dot{V}^m \tag{7}$$

$$\tilde{\omega}_{ik} = \Lambda_{[ik]} + \dot{V}_i V_k - \dot{V}_k V_i \tag{8}$$

$$\tilde{\sigma}_{ik} = \Lambda_{(ik)} - \frac{1}{3}\widetilde{\Theta}h_{ik} - 2C_{ikm}V^m - \dot{V}_i V_k - \dot{V}_k V_i \tag{9}$$

where $\dot{V}^i = V^i_{|k}V^k = \Lambda^i_k V^k$ and "|" denotes the Riemannian covariant derivative associated with the osculating Riemannian metric $a_{ij}(x) = g_{ij}(x, V(x))$. The symbols "[ ]," "( )" denote the antisymmetrization and symmetrization of Λ_{ik}, respectively. The tensor $C_{ijk} = \frac{1}{2}\frac{\partial f_{ij}(x,y)}{\partial y^k}$ is symmetric in all its subscripts. Therefore the first extended Finslerian covariant derivative of V can be expressed by

$$\Lambda_{ik} = \frac{1}{3}\widetilde{\Theta}h_{ik} + \tilde{\sigma}_{ik} + \tilde{\omega}_{ik} + \dot{V}_i V_k \tag{10}$$

The proper time derivative of any tensor T^{ij}_{kl} along the fluid flow lines can be given by

$$\dot{T}^{ij}_{kl} = T^{ij}_{kl;m}V^m$$

Remark *The consideration of a Finslerian incoherent fluid provides that the fluid lines are geodesics and* $\dot{V}^i = \Lambda^i_k V^k = 0$. *In this case the Finslerian geodesics coincide with the Riemannian ones of a V-Riemannian space (osculating Riemannian).*

In the following we derive the Raychauduri equation in a Finslerian spacetime. By the commutation relations of δ-covariant derivative of the vector field $V^i(x)$ we obtain

$$V_{i;hk} - V_{i;kh} = L^j_{ikh}V_j \tag{11}$$

where L^i_{jhk} curvature tensor is derived by the δ-*covariant derivative with respect to the* osculating affine connection coefficients $a^i_{jk}(x, V(x))$ (Asanov, 1985; Rund, 1959).

$$L^i_{jhk}(x, V(x)) = \left(\frac{\partial L^i_{jh}}{\partial x^k} + \frac{\partial L^i_{jh}}{\partial V^l}\frac{\partial V^l}{\partial x^k}\right) - \left(\frac{\partial L^i_{jk}}{\partial x^h} + \frac{\partial L^i_{jk}}{\partial V^l}\frac{\partial V^l}{\partial x^h}\right) + L^i_{mk}L^m_{jh} - L^i_{mh}L^m_{jk}$$

In virtue of (11) we get

$$V^k V_{i;hk} = L^i_{ihk}V_j V^k + V_{i;kh}V^k$$

or

$$V^k \Lambda_{ih;k} = L^j_{ihk}V_j V^k + \Lambda_{ik;h}V^k = L^j_{ihk}V_j V^k + (\Lambda_{ik}V^k)_{;h} - \Lambda_{ik}V^k_h$$

The last relation can be written as

$$V^k \Lambda_{ih;k} = -\Lambda_{ik}\Lambda^k_h + L^j_{ihk} V_j V^k + \dot{V}_{i;h} \tag{12}$$

Taking the trace of (12) we have

$$V^k (h^{i\ell}\Lambda_{i\ell})_{;k} = \frac{d\widetilde{\Theta}}{d\tau} = -\Lambda_{ik}\Lambda^{ik} - L_{i\ell} V^i V^\ell + \dot{V}^i_{;i} \tag{13}$$

Using the relations (7)–(10) and substituting in (13) we obtain

$$\frac{d\widetilde{\Theta}}{d\tau} = -\frac{1}{3}\widetilde{\Theta}^2 - \tilde{\sigma}_{ik}\tilde{\sigma}^{ik} + \tilde{\omega}_{ik}\tilde{\omega}^{ik} - L_{i\ell} V^i V^\ell + \dot{V}^i_{;i} \tag{14}$$

This is *Raychauduri's* equation of the Finslerian space-time. In the Riemannian approach of general relativity this equation plays a crucial role in the theorems of singularities. The change of expansion which is expressed by $\frac{d\widetilde{\Theta}}{d\tau}$ depends on the V-anisotropic behavior of Cartan tensor C^i_{jk} along the matter flow lines. When we consider an incoherent fluid, the fluid-lines are geodesics and the last term of right-hand side of (14) is $\dot{V}^i = 0$. In this case the Raychauduri equation is reduced to the form of a V-Riemannian metric space associated with the congruence of geodesics.

A perfect fluid in the Finslerian space time case has the form

$$T_{ij}(x, V(x)) = (\mu + p)V_i(x)V_j(x) + pa_{ij} \tag{15}$$

where $p = p(x)$, $\mu = \mu(x)$ represent the pressure and the density of the fluid respectively.

The Einstein equations can be written in the form

$$L_{ij}(x, V(x)) = K\left(T_{ij}(x, V(x)) - \frac{1}{2}T^k_k a_{ij}\right), \quad K : \text{constant} \tag{16}$$

where the Ricci tensor L_{ij} is directly determined by the matter energy–momentum tensor T_{ij} at each point, associated with the osculating Riemannian metric tensor $a_{ij}(x) = g_{ij}(x, V(x))$. Substitution of (15) to (16) gives

$$L_{ij} V^i V^j = \frac{1}{2} K(\mu + 3p) \tag{17}$$

The term $L_{i\ell} V^i V^\ell$ corresponds to an anisotropic gravitational influence of the matter along the world lines of the fluid and it expresses the tidal force of the field.

The form of Raychauduri equation in the case of perfect fluids (cf. 15) is given in virtue of (17) by

$$\dot{\tilde{\Theta}} = \frac{d\tilde{\Theta}}{d\tau} = -\frac{1}{3}\tilde{\Theta}^2 - \tilde{\sigma}_{ik}\tilde{\sigma}^{ik} + \tilde{\omega}_{ik}\tilde{\omega}^{ik} - \frac{1}{2}K(\mu + 3p) + \dot{V}^i_{;i} \tag{18}$$

The condition

$$L_{i\ell}V^i V^\ell > 0 \tag{19}$$

provides us with the so called strong energy condition for every time-like vector V^α tangent to time-like geodesics. From (17) and (19) we notice that the fluid energy μ and pressure p satisfy the energy condition $\mu + p > 0$. This condition uniquely defines the Finslerian world lines (congruences) of the fluid with $V(x)$ tangent vector field analogous to that of Riemannian framework (Ellis *et al.*, 1990). The term $L_{il}V^i V^l > 0$ can be considered as a key for the existence of conjugate points in the Finslerian space-time structure. Indeed if we are given a Finsler manifold (M^4, F) which is forward geodesically complete we may apply the theorems of Bonnet-Myers and Hawking–Ellis (Bao *et al.*, 2000; Hawking and Ellis, 1973) along flow lines of the fluid. By virtue of (17) and (19) we deduce

$$(\mu + 3p) \geq K' > 0 \tag{20}$$

where $K' = 6\lambda K^{-1}$, λ some constant. Then every geodesic with length $l = \frac{\pi}{\sqrt{K}}$ or longer must contain conjugate points.

3. ROBERTSON–WALKER METRIC IN A RANDERS–FINSLER SPACE-TIME

We define the Finslerian metric function (Stavrinos, 2002; Stavrinos and Diakogiannis, 2004) for an anisotropic model of the universe as

$$L(x, y) = \sqrt{\alpha_{ij} y^i y^j} + \phi(x)\bar{u}_a y^a \tag{21}$$

where $\bar{u}_a$ is a unit vector which expresses the observed anisotropy of the microwave background radiation, $\phi(x)$ plays the role of the "length" of the vector $u_a(x) = \phi(x)\bar{u}_a$, $\phi(x) \in \mathbb{R}$ .The coefficients g_{ij} in (21) we will are of general Riemannian type. In the following we use the Robertson–Walker model for the first term of (21) and we will derive the Ricci tensor, $L_{ij}(x, y)$, for a Finslerian space-time of constant curvature. Along to this direction we choose a coordinate system with spherical coordinates where the components $(t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ are functions of proper time τ. The Lagrangian of Riemannian geometry for the R–W metric

$$ds^2 = d\tau^2 - R^2(\tau)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

is defined by

$$L_{\mathrm{R}}(x^i, y^i) = \sqrt{\alpha_{ij} y^i y^j} = \frac{1}{2}\{\dot{t}^2 - R^2(t)[(1 - kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2]\}^{1/2} \tag{22}$$

where we set x^i as

$$x^i = \begin{pmatrix} t \\ r \\ \theta \\ \phi \end{pmatrix} \quad y^i = \begin{pmatrix} \dot{t} \\ \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} \tag{23}$$

$k = 0, \pm 1$ and $R(t)$ the scale factor. The R–W metric

$$\alpha_{ij} = diag\left(1, -\frac{R^2(t)}{1-kr^2}, -R^2(t)r^2, -R^2r^2\sin^2\theta\right)$$

In the work of Stavrinos and Diakogiannis (2004) we have considered a vector y^i to be a time-like or null vector. Under this condition the anisotropy vector $u_i = \phi(x)\hat{u}_i$ is space-like with components $u_i(x) = (0, u_1(x), u_2(x), u_3(x))$ and $u_\alpha(x) = u_a(t, r, \theta, \phi)$. Now we put $\sigma = \sqrt{\alpha_{ij} y^i y^j}$, $\beta = u_a(x)y^a$ in (21), thus we get

$$\sigma = \{\dot{t}^2 - R^2(t)[(1-kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2]\}^{1/2} \tag{24}$$

$$\beta = k_1\dot{r} + k_2\dot{\theta} + k_3\dot{\phi} \tag{25}$$

Next, we calculate the Finslerian metric which is derived by the generator metric function $F(x, y)$ in the form

$$g_{ij} = \frac{F}{\sigma}\alpha_{ij} + \frac{1}{\sigma}(y_i u_j + u_i y_j) - \frac{\beta}{\sigma^3} y_i y_j + u_i u_j. \tag{26}$$

The indices of y^i are raised and lowered by the metric g_{ij}. In such a case we have

$$y_i = \left(\dot{t}, -\frac{R^2\dot{r}}{1-kr^2}, -R^2r^2\dot{\theta}, -R^2r^2\sin^2\theta\,\dot{\phi}\right) \tag{27}$$

$$y_i u_j = \begin{pmatrix} 0 & \dot{t}u_1 & \dot{t}u_2 & \dot{t}u_3 \\ 0 & -\frac{R^2\dot{r}}{1-kr^2}u_1 & -\frac{R^2\dot{r}}{1-kr^2}u_2 & -\frac{R^2\dot{r}}{1-kr^2}u_3 \\ 0 & -R^2r^2\dot{\theta}u_1 & -R^2r^2\dot{\theta}u_2 & -R^2r^2\dot{\theta}u_3 \\ 0 & -R^2r^2\sin^2\theta\dot{\phi}u_1 & -R^2r^2\sin^2\theta\dot{\phi}u_2 & -R^2r^2\sin^2\theta\,\dot{\phi}u_3 \end{pmatrix} \tag{28}$$

Similarly we calculate

$$y_i y_j = \begin{pmatrix} \dot{t}^2 & -\frac{R^2\dot{r}\dot{t}}{1-kr^2} & -R^2r^2\dot{\theta}\dot{t} & -R^2r^2\sin^2\theta\,\dot{\phi}\dot{t} \\ -\frac{R^2\dot{r}\dot{t}}{1-kr^2} & \frac{R^4\dot{r}^2}{(1-kr^2)^2} & \frac{R^4r^2\dot{r}\dot{\theta}}{1-kr^2} & \frac{R^4r^2\sin^2\theta\,\dot{r}\dot{\phi}}{1-kr^2} \\ -R^2r^2\dot{\theta}\dot{t} & \frac{R^4r^2\dot{r}\dot{\theta}}{1-kr^2} & R^4r^4\dot{\theta}^2 & R^4r^4\sin^2\theta\,\dot{\theta}\dot{\phi} \\ -R^2r^2\sin^2\theta\,\dot{\phi}\dot{t} & \frac{R^4r^2\sin^2\theta\dot{r}\dot{\phi}}{1-kr^2} & R^4r^4\sin^2\theta\,\dot{\theta}\dot{\phi} & R^4r^4\sin^4\theta\,\dot{\phi}^2 \end{pmatrix} \tag{29}$$

Taking into account (24)–(30) we derive explicitly the metric g_{ij} by (21).

$$g_{ij} = \begin{pmatrix} f^1 & f^2 \\ f^3 & f^4 \end{pmatrix} \tag{30}$$

with

$$f^1 = \begin{pmatrix} f^1_{00} & f^1_{01} \\ f^1_{10} & f^1_{11} \end{pmatrix}$$

$$f^2 = \begin{pmatrix} f^2_{00} & f^2_{01} \\ f^2_{10} & f^2_{11} \end{pmatrix}$$

$$f^3 = \begin{pmatrix} f^3_{00} & f^3_{01} \\ f^3_{10} & f^3_{11} \end{pmatrix}$$

$$f^4 = \begin{pmatrix} f^4_{00} & f^4_{01} \\ f^4_{10} & f^4_{11} \end{pmatrix}$$

where

$$f^1_{00} = \frac{F}{\sigma} - \frac{\beta}{\sigma^3}\dot{t}^2$$

$$f^1_{01} = \frac{\dot{t}u_1}{\sigma} + \frac{\beta}{\sigma^3}\frac{R^2\dot{r}\dot{t}}{1-kr^2}$$

$$f^1_{10} = \frac{\dot{t}u_1}{\sigma} + \frac{\beta}{\sigma^3}\frac{R^2\dot{r}\dot{t}}{1-kr^2}$$

$$f^1_{11} = -\frac{F}{\sigma}\frac{R^2}{1-kr^2} - \frac{2R^2\dot{r}u_1}{\sigma(1-kr^2)} - \frac{\beta}{\sigma^3}\frac{R^4\dot{r}^2}{(1-kr^2)^2} + (u_1)^2$$

$$f^2_{00} = \frac{\dot{t}u_2}{\sigma} + \beta\sigma^{-3}R^2r^2\dot{\theta}\dot{t}$$

$$f^2_{01} = \dot{t}u_3\sigma^{-1} + \beta\sigma^{-3}R^2r^2\sin^2\theta\,\dot{\phi}\dot{t}$$

$$f^2_{10} = -\frac{R^2}{\sigma}\left[r^2\dot{\theta}u_1 + \frac{\dot{r}u_2}{1-kr^2}\right] - \frac{\beta}{\sigma^3}\frac{R^4r^2\dot{r}\dot{\theta}}{1-kr^2} + u_1u_2$$

$$f^2_{11} = -\frac{R^2}{\sigma}\left[r^2\sin^2\theta\,\dot{\phi}u_1 + \frac{\dot{r}u_3}{1-kr^2}\right] - \frac{\beta}{\sigma^3}\frac{R^4r^2\sin^2\theta\dot{r}\dot{\phi}}{1-kr^2} + u_1u_3$$

$$f^3_{00} = \frac{\dot{t}u_2}{\sigma} + \frac{\beta}{\sigma^3}R^2r^2\dot{\theta}\dot{t}$$

$$f_{01}^{3} = -\frac{R^2}{\sigma}\left[r^2\dot{\theta}u_1 + \frac{\dot{r}u_2}{1-kr^2}\right] - \frac{\beta}{\sigma^3}\frac{R^4r^2\dot{r}\dot{\theta}}{1-kr^2} + u_2u_1$$

$$f_{10}^{3} = \frac{\dot{t}u_3}{\sigma} + \frac{\beta}{\sigma^3}R^2r^2\sin^2\theta\dot{\phi}\dot{\tau}$$

$$f_{11}^{3} = -\frac{R^2}{\sigma}\left[r^2\sin^2\theta\dot{\phi}u_1 + \frac{\dot{r}u_3}{1-kr^2}\right] - \frac{\beta}{\sigma^3}\frac{R^4r^2\sin^2\theta\,\dot{r}\dot{\phi}}{1-kr^2} + u_3u_1$$

$$f_{00}^{4} = -\frac{F}{\sigma}R^2r^2 - \frac{2R^2r^2\dot{\theta}u_2}{\sigma} - \frac{\beta}{\sigma^3}R^4r^4\dot{\theta}^2 + (u_2)^2$$

$$f_{01}^{4} = -\frac{R^2r^2}{\sigma}[\sin^2\theta\dot{\phi}u_2 + \dot{\theta}u_3] - \frac{\beta}{\sigma^3}R^4r^4\sin^2\theta\dot{\theta}\dot{\phi} + u_2u_3$$

$$f_{10}^{4} = -\frac{R^2r^2}{\sigma}[\sin^2\theta\dot{\phi}u_2 + \dot{\theta}u_3] - \frac{\beta}{\sigma^3}R^4r^4\sin^2\theta\dot{\theta}\dot{\phi} + u_2u_3$$

$$f_{11}^{4} = -\frac{F}{\sigma}R^2r^2\sin^2\theta - \frac{2R^2r^2}{\sigma}\sin^2\theta\,\dot{\phi}u_3 - \frac{\beta}{\sigma^3}R^4r^4\sin^4\theta\dot{\phi}^2 + (u_3)^2$$

Thus the Ricci curvature L_{ij} in a Finslerian space-time of constant curvature

$$L_{ijkl} = K(g_{ik}g_{jl} - g_{il}g_{jk})$$

is explicitly given in virtue of (30) by

$$L_{ij} = 3Kf_{gj}$$

with

$$L_{ij} = L_{ijr}^{r}$$

The scalar curvature L is related with the constant K by the relation

$$L = 12K$$

Einstein's equation in this Finslerian space-time of constant curvature has the form

$$G_{ij} = kT_{ij}, \quad k = 8\pi$$

where T_{ij} is given by (15). Here we assume the cosmological constant $\Lambda = 0$.

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